A Fibrational Approach to Automata Theory

Liang-Ting Chen and Henning Urbat

Institut für Theoretische Informatik
Technische Universität Braunschweig, Germany

Abstract

For predual categories $\mathcal{C}$ and $\mathcal{D}$ we establish isomorphisms between opfibrations representing local varieties of languages in $\mathcal{C}$, local pseudovarieties of $\mathcal{D}$-monoids, and finitely generated profinite $\mathcal{D}$-monoids. The global sections of these opfibrations are shown to correspond to varieties of languages in $\mathcal{C}$, pseudovarieties of $\mathcal{D}$-monoids, and profinite equational theories of $\mathcal{D}$-monoids, respectively. As an application, we obtain a new proof of Eilenberg's variety theorem along with several related results, covering varieties of languages and their coalgebraic modifications, Straubing's C-varieties, fully invariant local varieties, etc., within a single framework.

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1 Introduction

In algebraic automata theory, regular languages are studied in connection with associated algebraic structures, using Eilenberg's celebrated variety theorem [7]. This theorem establishes a one-to-one correspondence between varieties of languages and pseudovarieties of monoids. By a *variety of languages* is meant a class of regular languages closed under the boolean operations (union, intersection and complement), left and right derivatives, and preimages under free monoid morphisms. A *pseudovariety of monoids* is a class of finite monoids closed under submonoids, quotients, and finite products.

Not every interesting class of languages falls within this scope. For this reason several authors weakened the closure properties in the definition of a variety of languages, and proved Eilenberg-type theorems for these modified varieties. For example, Pin's *positive varieties* [13], omitting closure under complement, correspond to pseudovarieties of *ordered* monoids. Polák's *disjunctive varieties* [15], further dropping closure under intersection, correspond to pseudovarieties of idempotent semirings. Reutenauer's *xor varieties* [17], closed under symmetric difference in lieu of the boolean operations, correspond to pseudovarieties of associative algebras over the field $\mathbb{Z}_2$. Straubing [19] introduced C-*varieties of languages*, where one restricts to closure under preimages of a chosen class C of free monoid morphisms in lieu of all free monoid morphisms. They are in bijection with C-*pseudovarieties of monoid morphisms*, these being classes of monoid morphisms with suitable closure properties.

A closely related line of work concerns "local" versions of Eilenberg's variety theorem, where languages over a fixed alphabet Σ are considered. Using the well-known duality between boolean algebras and Stone spaces, Pippenger [14] demonstrated that the boolean algebra $\mathsf{Reg}(\Sigma)$ of all regular languages over Σ dualises to the underlying Stone space of the free profinite monoid on Σ. Later, Gehrke, Grigorieff, and Pin [8] considered *local varieties of languages over* Σ, i.e. boolean subalgebras of $\mathsf{Reg}(\Sigma)$ closed under left and right derivatives, and characterised them as sets of regular languages over Σ definable by profinite equations.

In the recent work of Adámek, Milius, Myers, and Urbat [1, 2] a categorical approach to Eilenberg-type theorems was presented, covering many of the aforementioned results uniformly. The leading idea is to take two varieties of (possibly ordered) algebras $\mathcal{C}$ and $\mathcal{D}$

whose full subcategories of finite algebras are dually equivalent. Local varieties of languages are then modelled as coalgebras in $\mathcal{C}$, and monoids as monoid objects in $\mathcal{D}$. The main result of [1], the **General Local Variety Theorem**, states that *local varieties of languages over* Σ *in* $\mathcal{C}$ (= sets of regular languages over Σ closed under $\mathcal{C}$-algebraic operations and left and right derivatives) correspond to *local pseudovarieties of* Σ*-generated* $\mathcal{D}$*-monoids* (= sets of Σ-generated finite $\mathcal{D}$-monoids closed under quotients and subdirect products). The **General Variety Theorem** of [2] establishes a correspondence between *varieties of languages in* $\mathcal{C}$ and *pseudovarieties of* $\mathcal{D}$*-monoids*. Then the classical Eilenberg theorem is recovered by taking $\mathcal{C}$ = boolean algebras and $\mathcal{D}$ = sets, and other choices of $\mathcal{C}$ and $\mathcal{D}$ give its modifications due to Pin, Polák and Reutenauer along with new concrete Eilenberg-type correspondences.

The present paper is a continuation of the above work, aiming at two intriguing questions:

1. the connection between local pseudovarieties of $\mathcal{D}$-monoids and profinite $\mathcal{D}$-monoids;
2. the connection between the local and non-local versions of the General Variety Theorem;

left open in [1, 2]. To attack these questions, we organise all local varieties of languages into a category **LAN** whose objects are pairs (Σ, V) of a finite alphabet Σ and a local variety of languages over Σ in $\mathcal{C}$. With a suitable choice of morphisms in **LAN** (see Definition 3.7) the projection functor $p\colon \mathbf{LAN} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ into the category of finitely generated free $\mathcal{D}$-monoids, mapping (Σ, V) to the free $\mathcal{D}$-monoid over Σ, is an opfibration. In a similar fashion one can form the category **LPV** of local pseudovarieties of $\mathcal{D}$-monoids and the category **PFMon** of finitely generated profinite $\mathcal{D}$-monoids, which again yield opfibrations over $\mathbf{Free}(\mathbf{Mon}\mathcal{D})$.

$$\begin{tikzcd}
\mathbf{LAN} \arrow[r, "\cong"] \arrow[dr, "p"'] & \mathbf{LPV} \arrow[r, "\cong"] \arrow[d, "q"] & \mathbf{PFMon} \arrow[dl, "q'"] \\
& \mathbf{Free}(\mathbf{Mon}\mathcal{D}) &
\end{tikzcd}$$

Then we make two crucial observations. Firstly, we show that the global sections (namely, right inverse functors) of the above opfibrations p, q and q' correspond precisely to varieties of languages in $\mathcal{C}$, pseudovarieties of $\mathcal{D}$-monoids and profinite equational theories of $\mathcal{D}$-monoids, respectively. Secondly, we prove that the three opfibrations are isomorphic. The isomorphism $\mathbf{LAN} \cong \mathbf{LPV}$ is essentially the General Local Variety Theorem of [1], and the isomorphism $\mathbf{LPV} \cong \mathbf{PFMon}$ is based on a limit construction. From these isomorphisms it follows immediately that the global sections of our three opfibrations are in bijective correspondence:

> *There is a bijective correspondence between (i) varieties of languages in* $\mathcal{C}$*, (ii) pseudo-varieties of* $\mathcal{D}$*-monoids and (iii) profinite equational theories of* $\mathcal{D}$*-monoids.*

The bijection (ii)$\leftrightarrow$(iii) amounts to a categorical presentation of the well-known Reiterman-Banaschewski theorem [16, 5]. And (i)$\leftrightarrow$(ii) gives a conceptually completely different categorical proof of the General Variety Theorem in [2]. Furthermore, the flexibility of our fibrational setting leads rather easily to a number of additional results. For example, by replacing the category $\mathbf{Free}(\mathbf{Mon}\mathcal{D})$ with an arbitrary subcategory $\mathsf{C} \hookrightarrow \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ we obtain a generalised version of Straubing's variety theorem for C-varieties of languages, as well as a new local variety theorem for *fully invariant local varieties of languages*, i.e. local varieties closed under preimages of endomorphisms of free monoids.

Beyond these concrete results, we believe that the main contribution of the present paper is a further illumination of the intrinsic duality deeply hidden in algebraic language theory, most notably of the subtle interweavings of local and non-local structures, and the role of profinite theories.

2 Preliminaries

In this section we review the categorical approach to algebraic automata theory developed in [1, 2]. The idea is to interpret local varieties of languages inside a variety of algebras $\mathcal{C}$, and to relate them to finite monoids in another variety of (possibly ordered) algebras $\mathcal{D}$ which is **predual** to $\mathcal{C}$. The latter means that the full subcategories $\mathcal{C}_f$ and $\mathcal{D}_f$ of finite algebras are dually equivalent. Note that by an **ordered algebra** we mean an algebra (over a finitary signature Γ) with a poset structure on its underlying set making all operations monotone. Morphisms of ordered algebras are order-preserving Γ-homomorphisms. A *variety of ordered algebras* is a class of ordered algebras specified by inequalities $t_1 \leq t_2$ between Γ-terms.

▶ **Assumptions 2.1.** In the following $\mathcal{C}$ and $\mathcal{D}$ are predual varieties of algebras, where $\mathcal{D}$-algebras may be ordered, subject to the following conditions:

1. $\mathcal{C}$ and $\mathcal{D}$ are **locally finite**, i.e. every free algebra on a finite set is finite;
2. epimorphisms in $\mathcal{D}$ are surjective;
3. $\mathcal{D}$ is **entropic**, i.e. given an m-ary operation σ and an n-ary operation τ in the signature of $\mathcal{D}$ and variables x_{ij} ($i = 1, \ldots, m$, $j = 1, \ldots, n$), the following equation holds in $\mathcal{D}$:

$$\sigma(\tau(x_{11}, \ldots, x_{1n}), \ldots, \tau(x_{m1}, \ldots, x_{mn})) = \tau(\sigma(x_{11}, \ldots, x_{m1}), \ldots, \sigma(x_{1n}, \ldots, x_{mn})).$$

▶ **Notation 2.2.** We write $\Phi\colon \mathbf{Set} \to \mathcal{C}$ and $\Psi\colon \mathbf{Set} \to \mathcal{D}$ for the left adjoints to the forgetful functors $|-|\colon \mathcal{C} \to \mathbf{Set}$ and $|-|\colon \mathcal{D} \to \mathbf{Set}$, respectively. By $\mathbf{1}_{\mathcal{C}} = \Phi\mathbf{1}$ and $\mathbf{1}_{\mathcal{D}} = \Psi\mathbf{1}$ denote the free algebras over the singleton set.

▶ **Example 2.3.** The following pairs of varieties $\mathcal{C}/\mathcal{D}$ satisfy our assumptions. The details of the first three examples can be found in [11].

1. $\mathbf{BA}/\mathbf{Set}$: The Stone Representation Theorem exhibits a dual equivalence between the categories of finite boolean algebras and finite sets. It assigns to any finite boolean algebra B the set $\mathbf{BA}(B, \mathbf{2})$ of all homomorphisms into the two-chain $\mathbf{2}$. The dual of $h\colon A \to B$ is given by precomposition with h, i.e. $f \in \mathbf{BA}(B, \mathbf{2})$ is mapped to $f \circ h \in \mathbf{BA}(A, \mathbf{2})$.
2. $\mathbf{DLat}/\mathbf{Pos}$: Similarly, the Birkhoff Representation Theorem exhibits a dual equivalence between the categories of finite distributive lattices with 0 and 1 and finite posets. It assigns to a finite distributive lattice L the poset $\mathbf{DLat}(L, \mathbf{2})$, ordered pointwise, where $\mathbf{2}$ is the two-chain. On morphisms the dual equivalence again acts by precomposition.
3. $\mathbf{SLat}/\mathbf{SLat}$: The category of finite semilattices with 0 is self-dual: the dual equivalence maps a finite semilattice S to the semilattice $\mathbf{SLat}_f(S, \mathbf{2})$ whose join is taken pointwise.
4. $\mathbb{Z}_2$-$\mathbf{Vec}/\mathbb{Z}_2$-$\mathbf{Vec}$: The category of finite-dimensional vector spaces over any field F is self-dual, by mapping a vector space V to its dual space F-$\mathbf{Vec}(V, F)$. By restricting F to the binary field $\mathbb{Z}_2$, the category is also locally finite.

▶ **Remark 2.4.** Given a small finitely complete and cocomplete category $\mathcal{A}$ we denote by $\mathcal{Y}\colon \mathcal{A} \to \mathsf{Ind}\mathcal{A}$ and $\mathcal{Y}^{op}\colon \mathcal{A} \to \mathsf{Pro}\mathcal{A}$ the ind- and pro-completion of $\mathcal{A}$, i.e. the free completion under filtered colimits and cofiltered limits, respectively. There is an adjunction $F \dashv U\colon \mathsf{Pro}\mathcal{A} \to \mathsf{Ind}\mathcal{A}$ such that $\mathcal{Y}^{op} = F \circ \mathcal{Y}$ and $\mathcal{Y} = U \circ \mathcal{Y}^{op}$.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, "\mathcal{Y}"'] \arrow[dr, "\mathcal{Y}^{op}"] & \\
\mathsf{Ind}\mathcal{A} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathsf{Pro}\mathcal{A} \arrow[ll, bend left, "U"]
\end{tikzcd}$$

Applying this to $\mathcal{A} = \mathcal{C}_f$ with $\mathsf{Ind}(\mathcal{C}_f) = \mathcal{C}$ and $\mathsf{Pro}(\mathcal{C}_f) = \mathsf{Ind}(\mathcal{C}_f^{op})^{op} \cong \mathsf{Ind}(\mathcal{D}_f)^{op} = \mathcal{D}^{op}$, we see that the equivalence $\mathcal{C}_f \cong \mathcal{D}_f^{op}$ extends to an adjunction between $\mathcal{C}$ and $\mathcal{D}^{op}$. We denote both the equivalence $\mathcal{C}_f \cong \mathcal{D}_f^{op}$ and the induced adjunction between $\mathcal{C}$ and $\mathcal{D}^{op}$ by

$$S \dashv P \colon \mathcal{D}_f^{op} \xrightarrow{\cong} \mathcal{C}_f \quad \text{and} \quad S \dashv P \colon \mathcal{D}^{op} \to \mathcal{C}.$$

2.1 Local varieties of languages in $\mathcal{C}$

The coalgebraic treatment of automata roots in the observation that a deterministic automaton without an initial state is a coalgebra $\gamma = \langle \gamma^{1\text{st}}, \gamma^{2\text{nd}} \rangle \colon Q \to \mathbf{2} \times Q^\Sigma$ for the set functor $T_\Sigma^0 = \mathbf{2} \times (-)^\Sigma$. Here Σ is the finite input alphabet, $\mathbf{2} := \{\mathtt{yes}, \mathtt{no}\}$, $\gamma^{1\text{st}} \colon Q \to \mathbf{2}$ is the characteristic function of the final states, and $\gamma^{2\text{nd}} \colon Q \to Q^\Sigma$ is the transition map. In the following we consider automata in the category $\mathcal{C}$, which requires to replace the set $\mathbf{2}$ by a suitable "output" object in $\mathcal{C}$. Observe that the dual adjunction $S \dashv P \colon \mathcal{D}^{op} \to \mathcal{C}$ has *dualising objects* $O_\mathcal{C} := P\mathbf{1}_\mathcal{D}$ and $O_\mathcal{D} := S\mathbf{1}_\mathcal{C}$, that is, for all $M \in \mathcal{D}$ and $Q \in \mathcal{C}$ we have

$$|PM| \cong \mathcal{C}(\mathbf{1}_\mathcal{C}, PM) \cong \mathcal{D}(M, O_\mathcal{D}) \quad \text{and} \quad |SQ| \cong \mathcal{D}(\mathbf{1}_\mathcal{D}, SQ) \cong \mathcal{C}(Q, O_\mathcal{C}).$$

Taking $M = \mathbf{1}_\mathcal{D}$ we see that the set $|O_\mathcal{C}|$ is isomorphic to $|O_\mathcal{D}|$. Note that in each of the categories $\mathcal{C}/\mathcal{D}$ in Example 2.3 the objects $O_\mathcal{C}$ and $O_\mathcal{D}$ have a two-element carrier. Motivated by this observation, we replace the set $\mathbf{2}$ by the object $O_\mathcal{C}$ to define automata in $\mathcal{C}$.

▶ **Definition 2.5.** A **Σ-automaton in $\mathcal{C}$** is a coalgebra $\gamma = \langle \gamma^{1\text{st}}, \gamma^{2\text{nd}} \rangle \colon Q \to O_\mathcal{C} \times Q^\Sigma$ for the endofunctor $T_\Sigma := O_\mathcal{C} \times (-)^\Sigma$ on $\mathcal{C}$, where $(-)^\Sigma$ is the Σ-fold product. A **subautomaton** of (Q, γ) is a subcoalgebra of (Q, γ), represented by an injective coalgebra homomorphism into Q. An automaton is called **finite** if the object Q of states is finite, and **locally finite** if it is a filtered colimit of finite Σ-automata. The **rational fixpoint** ρT_Σ is the filtered colimit of all finite Σ-automata. The categories of Σ-automata, finite Σ-automata and locally finite Σ-automata in $\mathcal{C}$ are denoted by $\mathbf{Aut}\Sigma$, $\mathbf{Aut}_f\Sigma$ and $\mathbf{Aut}_{lf}\Sigma$, respectively. Their morphisms are coalgebra homomorphisms.

In [12, 3] it is shown that the rational fixpoint ρT_Σ is the terminal locally finite coalgebra (i.e. the terminal object of $\mathbf{Aut}_{lf}\Sigma$), with the structure map $\rho T_\Sigma \xrightarrow{\zeta} T_\Sigma(\rho T_\Sigma)$ an isomorphism. The rational fixpoint of the set functor $T_\Sigma^0 = \mathbf{2} \times (-)^\Sigma$ is the automaton of regular languages: the states of ρT_Σ^0 form the set $\mathsf{Reg}(\Sigma)$ of regular languages over Σ, the final states are those languages containing the empty word ε, and the transitions are given by left derivatives, that is, $L \xrightarrow{a} a^{-1}L = \{\, w \in \Sigma^* \mid aw \in L \,\}$ for $L \in \mathsf{Reg}(\Sigma)$ and $a \in \Sigma$.

▶ **Remark 2.6.** To simplify the presentation, we assume in the following that $|O_\mathcal{C}| = |O_\mathcal{D}| = \mathbf{2}$. The main reason is that in this case the rational fixpoint ρT_Σ is a lifting of the above automaton of regular languages to $\mathcal{C}$, see the next proposition. Without this assumption one needs to replace regular languages by regular behaviors, i.e. functions $\Sigma^* \to |O_\mathcal{C}|$ realised by finite Moore automata with output set $|O_\mathcal{C}|$. See also the discussion in [2, Section V].

▶ **Proposition 2.7** (see [1]). *The rational fixpoint ρT_Σ is carried by the set* $\mathsf{Reg}(\Sigma)$*. Its coalgebra structure $\rho T_\Sigma \xrightarrow{\zeta} O_\mathcal{C} \times (\rho T_\Sigma)^\Sigma$ is given by the $\mathcal{C}$-morphisms*

$$\zeta^{1\text{st}}(L) = \begin{cases} \mathtt{yes} & \text{if } \varepsilon \in L; \\ \mathtt{no} & \text{otherwise,} \end{cases} \quad \text{and} \quad \zeta^{2\text{nd}}(L)(a) = a^{-1}L.$$

In the light of this proposition we also write $\mathsf{Reg}(\Sigma)$ for the rational fixpoint ρT_Σ.

▶ **Example 2.8.** For $\mathcal{C} = \mathbf{BA}$, the rational fixpoint of T_Σ is the boolean algebra $\mathsf{Reg}(\Sigma)$ (w.r.t $\cup$, $\cap$, $(-)^{\complement}$, $\emptyset$ and Σ^*), endowed with the automata structure given by the boolean homomorphisms $\zeta^{1\text{st}}$ and $\zeta^{2\text{nd}}$. Similarly, for the other categories $\mathcal{C}$ of Example 2.3 the algebraic structure of $\rho T_\Sigma = \mathsf{Reg}(\Sigma)$ is *a)* $\cup$, $\cap$, $\emptyset$, and Σ^* for $\mathcal{C} = \mathbf{DLat}$; *b)* $\cup$ and $\emptyset$ for $\mathcal{C} = \mathbf{SLat}$; *c)* symmetric difference $L \oplus L' = (L \setminus L') \cup (L' \setminus L)$ and $\emptyset$ for $\mathcal{C} = \mathbb{Z}_2\text{-}\mathbf{Vec}$.

▶ **Definition 2.9.** A **local variety of languages over Σ in $\mathcal{C}$** is a subautomaton V of ρT_Σ closed under right derivatives, i.e. $L \in |V|$ implies $La^{-1} = \{\, w \in \Sigma^* \mid wa \in L \,\} \in |V|$ for all $a \in \Sigma$. The $\bigcap$-semilattices of all (finite) local varieties of languages over Σ in $\mathcal{C}$ are denoted by $\mathbf{LAN}^f_\Sigma$ and $\mathbf{LAN}_\Sigma$, respectively.

Observe that a local variety of languages is closed under(i) the $\mathcal{C}$-algebraic operations of ρT_Σ, being a sub*algebra* of ρT_Σ in $\mathcal{C}$, and (ii) left derivatives, being a sub*coalgebra* of ρT_Σ. For $\mathcal{C} = \mathbf{DLat}$ ($\mathcal{C} = \mathbf{BA}$) a local variety of languages is precisely a *(boolean) quotienting algebra of languages* in the sense of Gehrke et al. [8]: a set of regular languages over Σ closed under union, intersection (and complement) as well as left and right derivatives.

2.2 $\mathcal{D}$-monoids

Every entropic variety $\mathcal{D}$ of (ordered) algebras can be equipped with a symmetric monoidal closed structure $(\mathcal{D}, \otimes, \mathbf{1}_\mathcal{D})$, see [4] and [6, Theorem 3.10.1]. The unit $\mathbf{1}_\mathcal{D}$ is the free one-generated algebra and $\otimes$ is the usual tensor product of algebras, giving rise to a natural bijection between morphisms and bimorphisms in $\mathcal{D}$:

$$\mathsf{Hom}(A \otimes B, C) \cong \mathsf{Bihom}(A \times B, C).$$

Recall that a **bimorphism** $f\colon A \times B \to C$ in $\mathcal{D}$ is a set-theoretic function from $A \times B$ to C such that $f(a,-)\colon B \to C$ and $f(-,b)\colon A \to C$ are $\mathcal{D}$-morphisms for any $a \in A$ and $b \in B$.

Since the tensor product represents bimorphisms, the monoid objects of the monoidal category $(\mathcal{D}, \otimes, \mathbf{1}_\mathcal{D})$ correspond to the following algebraic concept:

▶ **Definition 2.10.** A **$\mathcal{D}$-monoid** $(M, \bullet, e)$ is an object M of $\mathcal{D}$ equipped with a monoid structure $(|M|, \bullet, e)$ in $\mathbf{Set}$ whose multiplication $\bullet\colon M \times M \to M$ is a $\mathcal{D}$-bimorphism. By a **morphism** $f\colon (M, \bullet, e) \to (M', \bullet', e')$ of $\mathcal{D}$-monoids is meant a morphism $f\colon M \to M'$ of $\mathcal{D}$ that is also a monoid morphism between the underlying monoids in $\mathbf{Set}$. By $\mathbf{Mon}_f\mathcal{D}$ and $\mathbf{Mon}\mathcal{D}$ we denote the categories of (finite) $\mathcal{D}$-monoids and all $\mathcal{D}$-monoid morphisms.

▶ **Example 2.11.** For the categories $\mathcal{D} = \mathbf{Set}$, $\mathbf{Pos}$, $\mathbf{SLat}$ and $\mathbb{Z}_2\text{-}\mathbf{Vec}$ of Example 2.3, the $\mathcal{D}$-monoids are precisely ordinary monoids, ordered monoids, idempotent semirings (with 0 and 1) and associative algebras over the field $\mathbb{Z}_2$, respectively.

▶ **Remark 2.12.** **1.** In $\mathcal{D}$ we choose the factorisation system (epi, strong mono). Recall that epimorphisms in $\mathcal{D}$ are precisely the surjective morphisms by Assumption 2.1.2. Strong monomorphisms are precisely the injective morphisms if $\mathcal{D}$ is a variety of algebras, and embeddings i.e. injective order-reflecting morphisms if $\mathcal{D}$ is a variety of ordered algebras. Hence every $\mathcal{D}$-morphism $f\colon A \to B$ factorises as $A \xrightarrow{\mathrm{Im}(f)} f[A] \xrightarrow{i} B$ where $\mathrm{Im}(f)$ is the restriction of f to the image $f[A]$ and i is injective (and order-reflecting). Further, the factorisation system has the **fill-in property**: given a surjective morphism e, an injective (and order-reflecting) morphism m and two morphisms u, v with $ue = mv$, there is a unique morphism d such that $u = md$ and $v = de$.

2. The factorisation system of $\mathcal{D}$ lifts to $\mathbf{Mon}\mathcal{D}$. Hence **submonoids** are represented by injective (order-reflecting) $\mathcal{D}$-monoid morphisms, and **quotient monoids** by surjective $\mathcal{D}$-monoid morphisms.

Since $\mathbf{Mon}\mathcal{D}$ is a variety of (ordered) algebras, the forgetful functor $\mathbf{Mon}\mathcal{D} \to \mathbf{Set}$ has a left adjoint constructing free $\mathcal{D}$-monoids. Here is a concrete construction:

▶ **Proposition 2.13** (see [1]). *The free $\mathcal{D}$-monoid on a set Σ is carried by the $\mathcal{D}$-object $\Psi\Sigma^*$. The monoid multiplication $\bullet$ extends the concatenation of words in Σ^*, and the unit is ϵ.*

A **finite Σ-generated $\mathcal{D}$-monoid** is a finite quotient $e_M : \Psi\Sigma^* \twoheadrightarrow M$ of the free $\mathcal{D}$-monoid on Σ. Given another finite Σ-generated $\mathcal{D}$-monoid $e_N : \Psi\Sigma^* \twoheadrightarrow N$ we write $M \le N$ if there is a $\mathcal{D}$-monoid morphism $f : N \to M$ satisfying $e_M = fe_N$. With respect to this order all (isomorphism classes of) finite Σ-generated $\mathcal{D}$-monoids form a poset $\mathsf{Quo}_f(\Psi\Sigma^*)$. Observe that $\mathsf{Quo}_f(\Psi\Sigma^*)$ is a join-semilattice: the join of M and N is the **subdirect product**, viz. the image of the morphism $\langle e_M, e_N\rangle : \Psi\Sigma^* \to M \times N$ given by

$$M \vee N := \{\, (e_M(x), e_N(x)) \in M \times N \mid x \in \Psi\Sigma^* \,\}.$$

▶ **Definition 2.14.** A **local pseudovariety of $\mathcal{D}$-monoids over Σ** is an *ideal* of $\mathsf{Quo}_f(\Psi\Sigma^*)$, i.e. a set of finite Σ-generated $\mathcal{D}$-monoids closed under quotients and subdirect products. By $\mathbf{LPV}_\Sigma$ we denote the $\bigcap$-semilattice of local pseudovarieties of $\mathcal{D}$-monoids over Σ.

▶ **Theorem 2.15** (General Local Variety Theorem [1]). *For each finite alphabet Σ,*

$$\mathbf{LAN}^f_\Sigma \cong \mathsf{Quo}_f(\Psi\Sigma^*) \quad \textit{and} \quad \mathbf{LAN}_\Sigma \cong \mathbf{LPV}_\Sigma.$$

▶ **Remark 2.16.** 1. The first isomorphism takes a finite local variety $O_\mathcal{C} \xleftarrow{\gamma^{1\text{st}}} V \xrightarrow{\gamma^{2\text{nd}}} V^\Sigma$ in $\mathcal{C}$ and applies the equivalence functor $S : \mathcal{C}_f \xrightarrow{\cong} \mathcal{D}^{op}_f$ to its coalgebra structure. This yields an algebra $\mathbf{1}_\mathcal{D} \cong S(O_\mathcal{C}) \xrightarrow{S\gamma^{1\text{st}}} SV \xleftarrow{S\gamma^{2\text{nd}}} S(V^\Sigma) \cong \coprod_\Sigma SV$ for the functor $F_\Sigma = 1_\mathcal{D} + \coprod_\Sigma(-)$ on $\mathcal{D}$. Since the free $\mathcal{D}$-monoid $\Psi\Sigma^*$ also carries the initial algebra for F_Σ, there is a unique F_Σ-algebra homomorphism $e_{SV} : \Psi\Sigma^* \to SV$ into the algebra constructed above. One then shows that e_{SV} is surjective and there is a unique $\mathcal{D}$-monoid structure on SV making e_{SV} a $\mathcal{D}$-monoid morphism. We call $e_{SV} : \Psi\Sigma^* \twoheadrightarrow SV$ the **(finite Σ-generated) $\mathcal{D}$-monoid corresponding to V**.
2. The second isomorphism follows immedatiely from the observation that $\mathbf{LAN}_\Sigma$ is isomorphic to the ideal completion of $\mathbf{LAN}^f_\Sigma$. Indeed, every finite local variety of languages is a compact element of $\mathbf{LAN}_\Sigma$, and every local variety is the directed union of its finite local subvarieties. Hence the isomorphism $\mathbf{LAN}_\Sigma \cong \mathbf{LPV}_\Sigma$ maps a local variety of languages $V \hookrightarrow \rho T_\Sigma$ to the local pseudovariety of all finite Σ-generated $\mathcal{D}$-monoids that correspond to some finite local subvariety of V. The inverse isomorphism maps a local pseudovariety P of $\mathcal{D}$-monoids over Σ to the directed union of all finite local varieties of languages in $\mathcal{C}$ that correspond to some element of P.

2.3 Preimages under $\mathcal{D}$-monoid morphisms

Recall from Remark 2.6 that we assume $|O_\mathcal{C}| = |O_\mathcal{D}| = \mathbf{2}$. Hence a language $L \subseteq \Delta^*$ may be identified with a morphism $L : \Psi\Delta^* \to O_\mathcal{D}$ of $\mathcal{D}$, viz. the adjoint transpose of the characteristic function $\Delta^* \to |O_\mathcal{D}|$. Given this identification, the **preimage** of L under a $\mathcal{D}$-monoid morphism $f : \Psi\Sigma^* \to \Psi\Delta^*$ is the composite $Lf : \Psi\Sigma^* \to \Psi\Delta^* \to O_\mathcal{D}$. By the adjunction $S \dashv P : \mathcal{D}^{op} \to \mathcal{C}$, the morphism Pf is essentially the preimage function, because

$$|Pf| \cong \mathcal{D}(f, O_\mathcal{D}) : \mathcal{D}(\Psi\Delta^*, O_\mathcal{D}) \to \mathcal{D}(\Psi\Sigma^*, O_\mathcal{D}).$$

In [2] it was shown that $|Pf|$ restricts to a $\mathcal{C}$-morphism $f^{-1}\colon \mathsf{Reg}(\Delta) \to \mathsf{Reg}(\Sigma)$, taking any language $L\colon \Psi\Delta^* \to O_{\mathcal{D}}$ in $\mathsf{Reg}(\Delta)$ to its f-preimage. This observation makes the following definition evident:

▶ **Definition 2.17.** Let $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ be a $\mathcal{D}$-monoid morphism and V and W local varieties of languages over Σ and Δ, respectively. Then V is said to be **closed under *f*-preimages of languages in *W*** if Diagram 1 below commutes for some $\mathcal{C}$-morphism h.

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, tail] & V \arrow[d, tail] \\ \mathsf{Reg}(\Delta) \arrow[r, "f^{-1}"'] & \mathsf{Reg}(\Sigma) \end{tikzcd}$$

Diagram 1

$$\begin{tikzcd} \Psi\Sigma^* \arrow[r, "f"] \arrow[d, two heads, "e_M"'] & \Psi\Delta^* \arrow[d, two heads, "e_N"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

Diagram 2

Here is a dual characterisation of preimage closure:

▶ **Lemma 2.18** (see [2]). *In Definition 2.17 let V and W be finite, and let $e_M\colon \Psi\Sigma^* \twoheadrightarrow M$ and $e_N\colon \Psi\Delta^* \twoheadrightarrow N$ be the finite $\mathcal{D}$-monoids corresponding to V and W, respectively. Then Diagram 1 commutes iff Diagram 2 with $g = Sh$ commutes.*

3 Fibrations for Languages and Monoids

We are ready to present our fibrational setting for (local) varieties of languages in $\mathcal{C}$ and (local) pseudovarieties of $\mathcal{D}$-monoids. For general information on fibred categories the reader is referred to [10]. Let us briefly recall some basic vocabulary:

▶ **Definition 3.1.** Let $p\colon \mathcal{E} \to \mathcal{B}$ be a functor.

1. An object $X \in \mathcal{E}$ is **above** $I \in \mathcal{B}$ if $pX = I$, and similarly a morphism f in $\mathcal{E}$ is above a morphism u in $\mathcal{B}$ if $pf = u$. A morphism f is called **vertical** (over I) if it is above an identity map (above id_I).
2. The **fibre** over $I \in \mathcal{B}$ is the subcategory $\mathcal{E}_I$ of $\mathcal{E}$ whose objects are the objects of $\mathcal{E}$ above I and whose morphisms are the vertical morphisms over I.
3. A morphism $f\colon X \to Y$ of $\mathcal{E}$ is **opcartesian over** $u\colon I \to J$ in $\mathcal{B}$ if $pf = u$ and for every morphism $g\colon X \to Z$ in $\mathcal{E}$ above wu for $w\colon J \to pZ$, there is a unique morphism $h\colon Y \to Z$ above w with $g = hf$.
4. $p\colon \mathcal{E} \to \mathcal{B}$ is an **opfibration** over $\mathcal{B}$ if for every $X \in \mathcal{E}$ and $u\colon pX \to J$ in $\mathcal{B}$ there is an opcartesian morphism $f\colon X \to Y$ above u, called an **opcartesian lifting** of u.
5. Two opfibrations $p\colon \mathcal{E} \to \mathcal{B}$ and $p'\colon \mathcal{E}' \to \mathcal{B}$ are **isomorphic** is there is an isomorphism $i\colon \mathcal{E} \cong \mathcal{E}'$ preserving indices, that is, $p'i = p$.
6. A **global section** of an obfibration $p\colon \mathcal{E} \to \mathcal{B}$ is a functor $s\colon \mathcal{B} \to \mathcal{E}$ with $es = id$.
7. A **poset opfibration** is an opfibration such that each fibre $\mathcal{E}_I$ $(I \in \mathcal{B})$ is a poset.
8. A **$\mathcal{B}$-indexed poset** is a functor $\mathcal{H}\colon \mathcal{B} \to \mathbf{Pos}$.

All opfibrations we consider below are poset opfibrations. They are effectively interchangeable with indexed posets via the **Grothendieck construction**:

1. Given a poset opfibration $p\colon E \to \mathcal{B}$ one defines an indexed poset $\mathcal{H}_p\colon \mathcal{B} \to \mathbf{Pos}$ as follows. Note first that every $\mathcal{B}$-morphism $I \xrightarrow{u} J$ with an object X above I has a *unique* opcartesian lifting $X \xrightarrow{f} u^*X$ because $\mathcal{E}_J$ is a poset. Then $\mathcal{H}_p$ is defined by

$$I \mapsto \mathcal{E}_I \quad \text{and} \quad \left(I \xrightarrow{u} J\right) \mapsto \left(\mathcal{E}_I \xrightarrow{u^*} \mathcal{E}_J\right)$$

   where u^* maps X to u^*X.
2. Conversely, given an indexed poset $\mathcal{H}\colon \mathcal{B} \to \mathbf{Pos}$, define the **Grothendieck completion** of $\mathcal{H}$ to be the category $\int \mathcal{H}$ with

   objects (I, x) where $I \in \mathcal{B}$ and $x \in \mathcal{H}I$;
   morphisms $(I, x) \xrightarrow{u} (J, y)$ where $I \xrightarrow{u} J$ is a morphism in $\mathcal{B}$ with $\mathcal{H}u(x) \leq_{\mathcal{H}J} y$.

   Then the projection functor $p_{\mathcal{H}}\colon \int \mathcal{H} \to \mathcal{B}$ mapping (I, x) to I and $(I, x) \xrightarrow{u} (J, y)$ to $I \xrightarrow{u} J$ is an opfibration.

The Grothendieck construction gives rise to an equivalence between suitable 2-categories of indexed posets and opfibrations. We only need the following weaker statement:

▶ **Theorem 3.2** (Grothendieck). *Every poset opfibration $p\colon \mathcal{E} \to \mathcal{B}$ is isomorphic to $p_{\mathcal{H}_p}\colon \mathcal{E} \to \mathcal{B}$, and every indexed poset $\mathcal{H}\colon \mathcal{B} \to \mathbf{Pos}$ is naturally isomorphic to $\mathcal{H}_{p_{\mathcal{H}}}\colon \mathcal{B} \to \mathbf{Pos}$. Furthermore, if $\mathcal{H}, \mathcal{H}'\colon \mathcal{B} \to \mathbf{Pos}$ are two naturally isomorphic indexed posets then $p_{\mathcal{H}}, p_{\mathcal{H}'}$ are isomorphic opfibrations.*

3.1 Local pseudovarieties of $\mathcal{D}$-monoids as an opfibration

In this section we organise the local pseudovarieties of $\mathcal{D}$-monoids into an opfibration $\mathbf{LPV} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$, or equivalently into an indexed poset $\mathbf{Free}(\mathbf{Mon}\mathcal{D}) \to \mathbf{Pos}$. The base category $\mathbf{Free}(\mathbf{Mon}\mathcal{D})$ is the category of finitely generated free $\mathcal{D}$-monoids: its objects are finite sets Σ, and its morphisms $\Sigma \xrightarrow{f} \Delta$ are all $\mathcal{D}$-monoid morphisms $\Psi\Sigma^* \xrightarrow{f} \Psi\Delta^*$ between the free $\mathcal{D}$-monoids on Σ and Δ, respectively. Hence $\mathbf{Free}(\mathbf{Mon}\mathcal{D})$ is dual to the Lawvere theory of the variety $\mathbf{Mon}\mathcal{D}$.

▶ **Definition 3.3.** The indexed poset $(-)_\sharp\colon \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \to \mathbf{Pos}$ is defined as follows:

1. To each finite set Σ it assigns the poset $\Sigma_\sharp = \mathbf{LPV}_\Sigma$ of all local pseudovarieties of $\mathcal{D}$-monoids over Σ, ordered by *reverse* inclusion $\supseteq$.
2. To each $\mathcal{D}$-monoid morphism $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ it assigns the monotone map $f_\sharp\colon \mathbf{LPV}_\Sigma \to \mathbf{LPV}_\Delta$, where for $P \in \mathbf{LPV}_\Sigma$ the local pseudovariety $f_\sharp(P) \in \mathbf{LPV}_\Delta$ consists of all finite Δ-generated $\mathcal{D}$-monoids N with $e_N f = g e_M$ for some $M \in P$ and some morphism g; see Diagram 2.

▶ **Lemma 3.4.** *$(-)_\sharp$ is a well-defined functor.*

The Grothendieck construction applied to the indexed poset $(-)_\sharp\colon \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \to \mathbf{Pos}$ yields the following equivalent opfibration:

▶ **Definition 3.5.** The category $\mathbf{LPV}$ of local pseudovarieties of $\mathcal{D}$-monoids has

objects (Σ, P) where P is a local pseudovariety of $\mathcal{D}$-monoids over Σ;
morphisms $(\Sigma, P) \xrightarrow{f} (\Delta, Q)$ where $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ is a $\mathcal{D}$-monoid morphism such that for every $N \in Q$ there exists $M \in P$ and $g\colon M \to N$ subject to Diagram 2.

The projection $\mathbf{LPV} \xrightarrow{q} \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ mapping (Σ, P) to Σ and $(\Sigma, P) \xrightarrow{f} (\Delta, Q)$ to f is called the **opfibration of local pseudovarieties of $\mathcal{D}$-monoids**.

3.2 Local varieties of languages in $\mathcal{C}$ as an opfibration

In complete analogy to Definition 3.3 and 3.5 we can define an indexed poset and its corresponding opfibration representing local varieties of languages in $\mathcal{C}$.

▶ **Definition 3.6.** The indexed poset $(-)_*\colon \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \to \mathbf{Pos}$ is defined as follows:

1. To each finite set Σ it assigns the poset $\Sigma_* = \mathbf{LAN}_\Sigma$ of all local varieties of languages over Σ in $\mathcal{C}$, ordered by *reverse* inclusion $\supseteq$.
2. To each $\mathcal{D}$-monoid morphism $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ it assigns the monotone map $f_*\colon \mathbf{LAN}_\Sigma \to \mathbf{LAN}_\Delta$, where for $V \in \mathbf{LAN}_\Sigma$ the local variety $f_*(V) \in \mathbf{LAN}_\Delta$ is the directed union of all local varieties W satisfying Diagram 1 for some h. In other words, $f_*(V)$ is the *largest* local variety of languages over Δ such that V is closed under f-preimages of languages in $f_*(V)$.

The Grothendieck construction gives the following opfibration:

▶ **Definition 3.7.** The category $\mathbf{LAN}$ of local varieties of languages in $\mathcal{C}$ has

objects (Σ, V) where V is a local variety of languages over Σ in $\mathcal{C}$;

morphisms $(\Sigma, V) \xrightarrow{f} (\Delta, W)$ where $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ is a $\mathcal{D}$-monoid morphism such that V is closed under f-preimages of languages in W.

The projection $\mathbf{LAN} \xrightarrow{p} \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ mapping (Σ, V) to Σ and $(\Sigma, V) \xrightarrow{f} (\Delta, W)$ to f is called the **opfibration of local varieties of languages in $\mathcal{C}$**.

The General Local Variety Theorem (see Theorem 2.15) implies that the two indexed posets $(-)_\sharp, (-)_*\colon \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \to \mathbf{Pos}$ of Definition 3.3 and 3.6 are naturally isomorphic. Indeed, recall from Remark 2.16 that the isomorphism $\mathbf{LPV}_\Sigma \cong \mathbf{LAN}_\Sigma$ sends a local pseudovariety $P \in \mathbf{LPV}_\Sigma$ to the directed union of all finite local varieties of languages over Σ in $\mathcal{C}$ corresponding to the finite Σ-generated $\mathcal{D}$-monoids in P. From this and Lemma 2.18 we conclude that the diagram below commutes for all $\mathcal{D}$-monoid morphisms $f\colon \Psi\Sigma^* \to \Psi\Delta^*$.

$$\begin{tikzcd}
\mathbf{LPV}_\Sigma \arrow[r, "\cong"] \arrow[d, "f_\sharp"'] & \mathbf{LAN}_\Sigma \arrow[d, "f_*"] \\
\mathbf{LPV}_\Delta \arrow[r, "\cong"'] & \mathbf{LAN}_\Delta
\end{tikzcd}$$

Hence, by Theorem 3.2, we get an isomorphism between the corresponding opfibrations:

▶ **Theorem 3.8.** *The opfibrations* $p\colon \mathbf{LAN} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ *and* $q\colon \mathbf{LPV} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ *are isomorphic.*

▶ **Definition 3.9.** By a **variety of languages in $\mathcal{C}$** is meant a global section of p, i.e. a functor $\mathcal{V}\colon \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \to \mathbf{LAN}$ with $p\mathcal{V} = id$.

In more concrete terms, a variety of languages in $\mathcal{C}$ is given by a collection of local varieties $V_\Sigma \in \mathbf{LAN}_\Sigma$ (where Σ ranges over all finite alphabets) such that for every $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ the local variety V_Σ is closed under f-preimages of languages in V_Δ. Varieties of languages in the categories $\mathcal{C} = \mathbf{BA}$, $\mathbf{DLat}$, $\mathbf{SLat}$ and $\mathbb{Z}_2$-$\mathbf{Vec}$ of Example 2.3 are precisely the classical varieties of languages of Eilenberg [7], the positive varieties of Pin [13], the disjunctive varieties of Polák [15] and the xor varieties of Reutenauer [17], respectively.

By Theorem 3.8 every global section of $p\colon \mathbf{LAN} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ corresponds uniquely to a global section of $q\colon \mathbf{LPV} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$. In the next section we will see that also the global sections of q admit a concrete interpretation.

4 Profinite $\mathcal{D}$-Monoids

A **profinite $\mathcal{D}$-monoid** is a cofiltered limit of finite $\mathcal{D}$-monoids, and the **profinite completion** $\widehat{M}$ of a $\mathcal{D}$-monoid M is the cofiltered limit of the diagram of all its finite quotients. Since limits in $\mathbf{Mon}\mathcal{D}$ are formed on the level of $\mathbf{Set}$, every profinite $\mathcal{D}$-monoid is equipped with a profinite topology, i.e. it can be viewed as a Stone space if $\mathcal{D}$ is a variety of algebras (or an ordered Stone space, if $\mathcal{D}$ is a variety of ordered algebras).[1] By $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$ denote the category of profinite $\mathcal{D}$-monoids with continuous (order-preserving) $\mathcal{D}$-monoid morphisms.

▶ **Theorem 4.1.** *1. $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$ is the pro-completion of the category $\mathbf{Mon}_f\mathcal{D}$ of finite $\mathcal{D}$-monoids (cf. Remark 2.4).*

2. The profinite completion $M \mapsto \widehat{M}$ gives a left adjoint to the forgetful functor $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D} \to \mathbf{Mon}\mathcal{D}$.

The first item follows from [11, Proposition VI.2.4]. The argument given there for varieties of algebras also applies to ordered algebras. The second item follows from a standard argument for ordinary monoids, see e.g., [18, Theorem 3.2.7].

▶ **Example 4.2.** For our predual categories $\mathcal{C}/\mathcal{D}$ of Example 2.3 we obtain the following descriptions of the categories $\mathsf{Pro}\mathcal{D}_f$, $\mathbf{Mon}\mathcal{D}$ and $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$, cf. [11, Corollary VI.2.4].

$\mathcal{C}$	$\mathcal{D}$	$\mathsf{Pro}\mathcal{D}_f$	$\mathbf{Mon}\mathcal{D}$	$\mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$
BA	**Set**	**Stone**	**Mon**	**Stone**(**Mon**)
DLat	**Pos**	**OStone**	**OMon**	(to be characterised)
SLat	**SLat**	**Stone**(**SLat**)	**ISRing**	**Stone**(**ISRing**)
$\mathbb{Z}_2$-**Vec**	$\mathbb{Z}_2$-**Vec**	**Stone**($\mathbb{Z}_2$-**Vec**)	$\mathbb{Z}_2$-**Alg**	**Stone**($\mathbb{Z}_2$-**Alg**)

Stone and **OStone** are the categories of (ordered) Stone spaces and continuous (order-preserving) maps. The categories in the fourth column are the categories of monoids, ordered monoids, idempotent semirings and $\mathbb{Z}_2$-algebras, respectively; see Example 2.11. By **Stone**($\mathcal{A}$) for a variety of algebras $\mathcal{A}$ we mean the category of $\mathcal{A}$-algebras in **Stone**. For example, **Stone**(**Mon**) is the category of monoids equipped with a Stone topology (making the monoid multiplication continuous) and continuous monoid morphisms.

4.1 Local pseudovarieties of $\mathcal{D}$-monoids vs. profinite $\mathcal{D}$-monoids

In this section we show how to identify local pseudovarieties of $\mathcal{D}$-monoids over Σ with Σ-generated profinite $\mathcal{D}$-monoids. In the following **quotients** of profinite $\mathcal{D}$-monoids are meant to be represented by surjective continuous $\mathcal{D}$-monoid morphisms. A **Σ-generated profinite $\mathcal{D}$-monoid** is a quotient of $\widehat{\Psi\Sigma^*}$, the profinite completion of the free $\mathcal{D}$-monoid $\Psi\Sigma^*$. Note that, by Theorem 4.1, $\widehat{\Psi\Sigma^*}$ is the free profinite $\mathcal{D}$-monoid on the free $\mathcal{D}$-monoid $\Psi\Sigma^*$ w.r.t. the forgetful functor $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D} \to \mathbf{Mon}\mathcal{D}$, and hence also the free profinite $\mathcal{D}$-monoid on the set Σ w.r.t. the composite forgetful functor $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D} \to \mathbf{Mon}\mathcal{D} \to \mathbf{Set}$. The following standard facts will be useful.

▶ **Lemma 4.3** (see e.g., [18, Chapter 3]). *Let $F\colon \mathcal{J} \to \mathbf{KHaus}$ be a cofiltered diagram in the category of compact Hausdorff spaces and continuous functions.*

[1] An **(ordered) Stone space** is a compact space such that for every $x \neq y$ (resp. $x \not\leq y$) there exists a clopen (upper) set containing x but not y.

1. *If every* $F_i \xrightarrow{Ff} F_j$ *for* $i \xrightarrow{f} j$ *is surjective, then the limit projections* $\operatorname{Lim} F \xrightarrow{\pi_i} F_i$ *are also surjective.*
2. *If* $\varphi\colon \Delta X \Rightarrow F$ *is a cone over* F *such that every projection* $\varphi_i\colon X \to F_i$ *is surjective, then the mediating morphism* $X \to \operatorname{Lim} F$ *is also surjective.*

▶ **Remark 4.4.** **1.** To each local pseudovariety $P \in \mathbf{LPV}_\Sigma$ we associate a Σ-generated profinite $\mathcal{D}$-monoid as follows. Note first that P defines a cofiltered diagram in $\mathsf{Pro}\mathbf{Mon}\mathcal{D}$ via the projection $(e\colon \Psi\Sigma^* \twoheadrightarrow M) \mapsto M$. Since the connecting morphisms are surjective, the above lemma implies that every limit projection $\operatorname{Lim} P \to M$ for $M \in P$ is surjective. Moreover, given $P \subseteq P'$ in $\mathbf{LPV}_\Sigma$, there is a surjective mediating morphism $h\colon \operatorname{Lim} P' \to \operatorname{Lim} P$. In particular, taking P' to be the local pseudovariety of *all* finite quotients of $\Psi\Sigma^*$ with $\operatorname{Lim} P' = \widehat{\Psi\Sigma^*}$ we get a surjective morphism $\widehat{\Psi\Sigma^*} \twoheadrightarrow \operatorname{Lim} P$, i.e. a Σ-generated profinite $\mathcal{D}$-monoid.

2. Conversely, to each Σ-generated profinite $\mathcal{D}$-monoid $e_\Sigma\colon \Psi\Sigma^* \twoheadrightarrow F\Sigma$ we associate a local pseudovariety $\mathcal{V}_{F\Sigma} \in \mathbf{LPV}_\Sigma$ as follows: $\mathcal{V}_{F\Sigma}$ consists of all finite Σ-generated $\mathcal{D}$-monoids of the form $\Psi\Sigma^* \xrightarrow{\eta} \widehat{\Psi\Sigma^*} \xrightarrow{e_\Sigma} F\Sigma \xrightarrow{e_M} M$, where η is the universal arrow of the adjunction between $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$ and $\mathbf{Mon}\mathcal{D}$ (see Theorem 4.1) and M is any finite quotient of $F\Sigma$. Observe that such a composite $e_M e_\Sigma \eta$ is always surjective: since $\widehat{\Psi\Sigma^*}$ is the limit of all finite quotients of $\Psi\Sigma^*$, and M is finite (hence a finitely copresentable object of $\mathsf{Pro}\mathbf{Mon}\mathcal{D}$), the morphism $e_M e_\Sigma$ factorises through some limit projection π_N, where N is a finite quotient of $\Psi\Sigma^*$:

$$\begin{tikzcd}
\Psi\Sigma^* \arrow[r, "\eta"] \arrow[dr, two heads] & \widehat{\Psi\Sigma^*} \arrow[r, two heads, "e_\Sigma"] \arrow[d, two heads, "\pi_N"'] & F\Sigma \arrow[d, two heads, "e_M"] \\
& N \arrow[r, two heads, "f"'] & M
\end{tikzcd}$$

It is not difficult to to see that the two constructions of Remark 4.4 are mutually inverse. More precisely:

▶ **Theorem 4.5.** *Let* Σ *be a finite set.*

1. *Every* Σ*-generated profinite* $\mathcal{D}$*-monoid* $F\Sigma$ *corresponds uniquely to a local pseudovariety* $\mathcal{V}_{F\Sigma}$ *of* $\mathcal{D}$*-monoids over* Σ*. That is,*

$$\mathsf{Quo}(\widehat{\Psi\Sigma^*}) \cong \mathbf{LPV}_\Sigma,$$

where $\mathsf{Quo}(\widehat{\Psi\Sigma^*})$ *denotes the poset of* Σ*-generated profinite* $\mathcal{D}$*-monoids.*

2. *Let* $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ *be a* $\mathcal{D}$*-monoid morphism,* $F\Sigma$ *a* Σ*-generated profinite* $\mathcal{D}$*-monoid and* $F\Delta$ *a* Δ*-generated profinite* $\mathcal{D}$*-monoid. Then the right-hand diagram below commutes for some* h *iff for every* $N \in \mathcal{V}_{F\Delta}$ *there is some* $M \in \mathcal{V}_{F\Sigma}$ *and a morphism* h_N *making the left-hand diagram commute:*

$$\begin{tikzcd}
\Psi\Sigma^* \arrow[r, "f"] \arrow[d, two heads] & \Psi\Delta^* \arrow[d, two heads] \\
M \arrow[r, "h_N"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\Psi\Sigma^*} \arrow[r, "\widehat{f}"] \arrow[d, two heads] & \widehat{\Psi\Delta^*} \arrow[d, two heads] \\
F\Sigma \arrow[r, "h"'] & F\Delta
\end{tikzcd}$$

From the opfibration $q\colon \mathbf{LPV} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ we thus get the following isomorphic opfibration:

▶ **Definition 4.6.** The category **PFMon** has

objects $(\Sigma, F\Sigma)$ where $F\Sigma$ is a Σ-generated profinite $\mathcal{D}$-monoid;

morphisms $(\Sigma, F\Sigma) \xrightarrow{f} (\Delta, F\Delta)$ where $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ is a $\mathcal{D}$-monoid morphism making the following diagram commute for some h:

$$\begin{tikzcd} \widehat{\Psi\Sigma^*} \arrow[r, "\widehat{f}"] \arrow[d, two heads] & \widehat{\Psi\Delta^*} \arrow[d, two heads] \\ F\Sigma \arrow[r, "h"'] & F\Delta \end{tikzcd} \tag{1}$$

The projection $\mathbf{PFMon} \xrightarrow{q'} \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ sending $(\Sigma, F\Sigma)$ to Σ and $(\Sigma, F\Sigma) \xrightarrow{f} (\Delta, F\Delta)$ to f is called the **opfibration of finitely generated profinite $\mathcal{D}$-monoids**.

For the record:

▶ **Corollary 4.7.** *The opfibrations $q\colon \mathbf{LPV} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ and $q'\colon \mathbf{PFMon} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ are isomorphic.*

4.2 Pseudovarieties of $\mathcal{D}$-monoids vs. profinite equational theories

By a **pseudovariety of $\mathcal{D}$-monoids** is meant a class of finite $\mathcal{D}$-monoids closed under submonoids, quotients and finite products. In this section we relate pseudovarieties of $\mathcal{D}$-monoids to profinite equational theories of $\mathcal{D}$-monoids.

▶ **Definition 4.8.** A **profinite equational theory** of $\mathcal{D}$-monoids is a global section $\mathcal{T}\colon \mathbf{Free}(\mathbf{Mon}\mathcal{D}) \to \mathbf{PFMon}$ of the opfibration $q'\colon \mathbf{PFMon} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$.

More explicitly, a profinite equational theory associates to each finite set Σ a Σ-generated profinite monoid $e_\Sigma\colon \widehat{\Psi\Sigma^*} \twoheadrightarrow F\Sigma$ such that, for all $f\colon \Psi\Sigma^* \to \Psi\Delta^*$, diagram (1) commutes for some h.

▶ **Remark 4.9.**
1. To each profinite equational theory $\mathcal{T}$ with $\mathcal{T}\Sigma = (\Sigma, F\Sigma)$ we associate a pseudovariety $\mathcal{V}$ of $\mathcal{D}$-monoids as follows: $\mathcal{V}$ consists of all finite $\mathcal{D}$-monoids M such that for all $\mathcal{D}$-monoid morphisms $f\colon \widehat{\Psi\Sigma^*} \to M$ there exists a (necessarily unique) $\mathcal{D}$-monoid morphism $\overline{f}\colon F\Sigma \to M$ with $\overline{f} e_\Sigma = f$.

$$\begin{tikzcd} \widehat{\Psi\Sigma^*} \arrow[r, two heads, "e_\Sigma"] \arrow[dr, "f"'] & F\Sigma \arrow[d, dashed, "\overline{f}"] \\ & M \end{tikzcd}$$

2. Conversely, to each pseudovariety $\mathcal{V}$ of $\mathcal{D}$-monoids we associate a profinite equational theory $\mathcal{T}$ with $\mathcal{T}\Sigma = (\Sigma, F\Sigma)$ as follows: given Σ, form the local pseudovariety P_Σ of all Σ-generated finite $\mathcal{D}$-monoids $e\colon \Psi\Sigma^* \twoheadrightarrow M$ with $M \in \mathcal{V}$. Then $F\Sigma$ is the Σ-generated profinite $\mathcal{D}$-monoid defined by P_Σ, see Remark 4.4 and Theorem 4.5.

   Again, these constructions are mutually inverse:

▶ **Theorem 4.10.** *The maps $\mathcal{T} \mapsto \mathcal{V}$ and $\mathcal{V} \mapsto \mathcal{T}$ define a bijective correspondence between profinite equational theories and pseudovarieties of $\mathcal{D}$-monoids.*

▶ **Remark 4.11.** This theorem can be viewed as a categorical presentation of the well-known Reiterman-Banaschewski correspondence [16, 5]. The difference lies in the definition of a profinite theory: Reiterman and Banaschewski work with profinite equations (i.e. pairs of elements of free profinite monoids) while we work with quotients of free profinite monoids.

5 Eilenberg-type Correspondences

Putting the results of our paper together we will now derive a number of Eilenberg-type theorems. Each of these theorems is an immediate consequence of the isomorphisms we established between our opfibrations p, q and q' (see the diagram in the Introduction) and the characterisation of their global sections. First, by Theorem 4.5 we get another version of the General Local Variety Theorem, i.e. Theorem 2.15).

▶ **Theorem 5.1** (General Local Variety Theorem II)**.** *There is a one-to-one correspondence between local varieties of languages over Σ in $\mathcal{C}$ and Σ-generated profinite $\mathcal{D}$-monoids:*

$$\mathbf{LAN}_\Sigma \cong \mathsf{Quo}(\widehat{\Psi\Sigma^*}).$$

Similarly, by Theorem 3.8, Corollary 4.7 and Theorem 4.10 we recover the main result of [2], where a completely different proof method was applied:

▶ **Theorem 5.2** (General Variety Theorem)**.** *There is a one-to-one correspondence between varieties of languages in $\mathcal{C}$ and pseudovarieties of $\mathcal{D}$-monoids.*

An interesting generalisation of this theorem emerges by restricting $\mathbf{Free}(\mathbf{Mon}\mathcal{D})$ to a subcategory. Recall that the pullback in $\mathbf{Cat}$ of an opfibration $p\colon \mathcal{E} \to \mathcal{B}$ along any functor $F\colon \mathcal{B}' \to \mathcal{B}$ is again an opfibration, see e.g., [10, Lemma 1.5.1].

▶ **Definition 5.3.** For a subcategory $\mathsf{C} \hookrightarrow \mathbf{Free}(\mathbf{Mon}\mathcal{D})$, a **$\mathsf{C}$-variety of languages in $\mathcal{C}$** is a global section of the opfibration $p_{\mathsf{C}}\colon \mathbf{LAN}_{\mathsf{C}} \to \mathsf{C}$ obtained as the pullback of the opfibration p along the inclusion. Similarly, a **profinite equational C-theory of $\mathcal{D}$-monoids** is a global section of the opfibration $q'_{\mathsf{C}}\colon \mathbf{PFMon}_{\mathsf{C}} \to \mathsf{C}$ obtained as the pullback of $q'\colon \mathbf{PFMon} \to \mathbf{Free}(\mathbf{Mon}\mathcal{D})$ along the inclusion.

$$\begin{tikzcd}
\mathbf{LAN}_{\mathsf{C}} \arrow[r, hook] \arrow[d, "p_{\mathsf{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{LAN} \arrow[d, "p"] \\
\mathsf{C} \arrow[r, hook] & \mathbf{Free}(\mathbf{Mon}\mathcal{D})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{PFMon}_{\mathsf{C}} \arrow[r, hook] \arrow[d, "q'_{\mathsf{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{PFMon} \arrow[d, "q'"] \\
\mathsf{C} \arrow[r, hook] & \mathbf{Free}(\mathbf{Mon}\mathcal{D})
\end{tikzcd}$$

More explicitly, a profinite equational C-theory associates to each $\Sigma \in \mathsf{C}$ a Σ-generated profinite monoid $e_\Sigma\colon \widehat{\Psi\Sigma^*} \twoheadrightarrow F\Sigma$ such that, for all $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ in C, diagram (1) commutes for some h. Similarly, a C-variety of languages determines a family $(V_\Sigma)_{\Sigma\in\mathsf{C}}$, where V_Σ is a local variety of languages over Σ in $\mathcal{C}$ and, for each $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ in C, the local variety V_Σ is closed under f-preimages of languages in V_Δ. For the case where $\mathcal{C} = \mathbf{BA}$, $\mathcal{D} = \mathbf{Set}$ and the subcategory C contains all objects of $\mathbf{Free}(\mathbf{Mon})$, this definition coincides with the concept of a C-variety of languages introduced by Straubing [19]. He also proved a special case of Theorem 5.4 below. Observe that since the opfibrations p and q' are isomorphic, so are their pullbacks p_{C} and q'_{C}. Therefore:

▶ **Theorem 5.4** (General Variety Theorem for C-varieties of languages)**.** *There is a one-to-one correspondence between C-varieties of languages in $\mathcal{C}$ and profinite equational C-theories of $\mathcal{D}$-monoids.*

As an application of this theorem, let us choose C to be the full subcategory of $\mathbf{Free}(\mathbf{Mon}\mathcal{D})$ on a single object Σ. Then a C-variety of languages in $\mathcal{C}$ is precisely a local variety of languages over Σ in $\mathcal{C}$ closed under preimages of $\mathcal{D}$-monoid endomorphisms $f\colon \Psi\Sigma^* \to \Psi\Sigma^*$. We call such a local variety **fully invariant**. A profinite equational C-theory consists of a single Σ-generated profinite $\mathcal{D}$-monoid $e\colon \widehat{\Psi\Sigma^*} \twoheadrightarrow F\Sigma$ such that, for all $\mathcal{D}$-monoid endomorphisms $f\colon \Psi\Sigma^* \to \Psi\Sigma^*$, $e\widehat{f}$ factors through e.

$$\begin{tikzcd}
\widehat{\Psi\Sigma^*} \arrow[r, "\widehat{f}"] \arrow[d, two heads, "e"'] & \widehat{\Psi\Sigma^*} \arrow[d, two heads, "e"] \\
F\Sigma \arrow[r, dashed] & F\Sigma
\end{tikzcd}$$

Again, such a Σ-generated profinite $\mathcal{D}$-monoid is called **fully invariant**. Hence full invariance means precisely that (in-)equalities are stable under translations, i.e. for every $x, y \in \widehat{\Psi\Sigma^*}$ and $f\colon \Psi\Sigma^* \to \Psi\Sigma^*$ we have that $e(x) = e(y)$ implies $e(\widehat{f}x) = e(\widehat{f}y)$; in respect that $\mathcal{D}$-algebras are ordered, $e(x) \leq e(y)$ implies $e(\widehat{f}x) \leq e(\widehat{f}y)$. Therefore Theorem 5.4 gives the following:

▶ **Theorem 5.5** (Local Variety Theorem for Fully Invariant Varieties). *There is a one-to-one correspondence between fully invariant local varieties over Σ in $\mathcal{C}$ and fully invariant Σ-generated profinite $\mathcal{D}$-monoids.*

6 Conclusions and Future Work

In this paper we studied varieties of languages, pseudovarieties of monoids and profinite equational theories from an abstract fibrational viewpoint. This led us to conceptually new proofs and generalisations for a number of Eilenberg-Reiterman-type results.

Our notion of profinite equational theory is introduced on a rather abstract level, and it would be helpful to characterise theories syntactically and compare them with classical developments [16, 5]. To this end one can observe that in the category of compact Hausdorff spaces every epimorphism is regular. Hence, if $\mathcal{D}$-algebras are non-ordered, every Σ-generated profinite $\mathcal{D}$-monoid $e\colon \widehat{\Psi\Sigma^*} \twoheadrightarrow M$ is the coequaliser of its kernel pair $\pi_1, \pi_2\colon E \rightrightarrows \widehat{\Psi\Sigma^*}$, where E is the kernel congruence defined by

$$E = \{\, (u, v) \in \widehat{\Psi\Sigma^*} \times \widehat{\Psi\Sigma^*} \mid e(u) = e(v) \,\}.$$

Hence a profinite equational theory corresponds to a family of *profinite equations*, i.e. pairs of elements of a free profinite monoid. From this observation it should be possible to obtain syntactic counterparts of our results, e.g., a generalisation of the main result of Gehrke et al. [8] that local varieties of languages in **BA** and **DLat** are definable by profinite identities.

In addition, it would be useful to develop a notion of *morphism* between profinite equational theories, and correspondingly between varieties of languages, hence lifting our generalised Eilenberg-Reiterman correspondences from an isomorphism of posets to an equivalence of categories. Such a result may further justify the importance of a categorical treatment of algebraic automata theory.

References

1 J. Adámek, S. Milius, R. S. Myers, and H. Urbat. Generalized Eilenberg Theorem I : Local Varieties of Languages. In A. Muscholl (ed.) *Found. Softw. Sci. Comput. Struct.* LNCS, vol. 8412, pp. 366–380. Springer Berlin Heidelberg, 2014. `arXiv:1501.02834 [cs.FL]`

2 J. Adámek, R. S. Myers, S. Milius, and H. Urbat. Varieties of Languages in a Category. Accepted for LICS 2015. `arXiv:1501.05180 [cs.FL]`

3 J. Adámek, S. Milius and J. Velebil. Iterative Algebras at Work. *Math. Structures Comput. Sci.*, 16 (6), 1085–1131, 2006.

4 B. Banaschewski and E. Nelson. Tensor products and bimorphisms. Canad. Math. Bull. 19, 385–402, 1976.

5 B. Banaschewski. The Birkhoff Theorem for varieties of finite algebras. *Algebr. universalis*, 17(1):360–368, 1983.

6 F. Borçeux. Handbook of Categorical Algebra: Volume 2, Categories and Structures. Cambridge University Press, 1994.

7 S. Eilenberg. *Automata, Languages, and Machines Vol. B*, Academic Press, New York, 1976.

8 M. Gehrke, S. Grigorieff, and J.-É. Pin. Duality and equational theory of regular languages. In *Autom. Lang. Program.* LNCS, vol. 5126, pp. 246–257. Springer Berlin Heidelberg, 2008.

9 C. Hermida and B. Jacobs. Structural induction and coinduction in a fibrational setting. *Inf. Comput.*, 145(2):107–152, 1998.

10 B. Jacobs. *Categorical Logic and Type Theory.* North Holland, Amsterdam, 1999.

11 P. T. Johnstone. *Stone spaces.* Cambridge University Press, 1982.

12 S. Milius. A sound and complete calculus for finite stream circuits. *25th Annu. IEEE Symp. Log. Comput. Sci.*, pp. 421–430, 2010.

13 J.-É. Pin. A variety theorem without complementation. *Russ. Math. (Iz. VUZ)*, 39:80–90, 1995.

14 N. Pippenger. Regular languages and stone duality. *Theory Comput. Syst.*, 30(2):121–134, 1997.

15 L. Polák. Syntactic semiring of a language. In J. Sgall, A. Pultr, and P. Kolman (eds.) *Math. Found. Comput. Sci.* LNCS, vol. 2136, pp. 611–620. Springer Berlin Heidelberg, 2001.

16 J. Reiterman. The Birkhoff theorem for finite algebras. *Algebr. Universalis*, 14(1):1–10, 1982.

17 C. Reutenauer. Séries formelles et algèbres syntactiques. *J. Algebr.*, 66(2):448–483, 1980.

18 J. Rhodes and B. Steinberg. *The q-theory of Finite Semigroups.* Springer US, 2009.

19 H. Straubing. On logical descriptions of regular languages. In S. Rajsbaum (ed.) *LATIN 2002 Theor. Informatics.* LNCS, vol. 2286, pp. 528–538. Springer Berlin Heidelberg, 2002.

A Ind-completion and pro-completion

The following facts on ind/pro-completions are standard results, see [11] for further detail.

▶ **Definition A.1.** **1.** An **ind-completion** of a small category $\mathcal{A}$ is a full and faithful functor $\mathcal{A} \rightarrowtail \mathsf{Ind}\mathcal{A}$ such that $\mathsf{Ind}\mathcal{A}$ has filtered colimits and every functor F from $\mathcal{A}$ to a category $\mathcal{B}$ with filtered colimits has an extension $\overline{F}\colon \mathsf{Ind}A \to B$ which preserves filtered colimits and is unique up to natural isomorphism:

$$\begin{tikzcd} \mathcal{A} \arrow[r, tail] \arrow[dr, "F"'] & \mathsf{Ind}\mathcal{A} \arrow[d, dashed, "\overline{F}"] \\ & \mathcal{B} \end{tikzcd}$$

If $\mathcal{A}$ is finitely cocomplete, then $\mathsf{Ind}\mathcal{A}$ is complete and cocomplete. In particular, every locally finite variety $\mathcal{D}$ is an ind-completion of its full subcategory $\mathcal{D}_f$ on finite algebras.

2. Dually a **pro-completion** of a small category $\mathcal{A}$ is a full and faithful functor $\mathcal{A} \rightarrowtail \mathsf{Pro}\mathcal{A}$ such that $\mathsf{Pro}\mathcal{A}$ has cofiltered limits and every functor F from $\mathcal{A}$ to a category $\mathcal{B}$ with cofiltered limits has an extension $\overline{F}\colon \mathsf{Pro}A \to B$ which preserves cofiltered limits and is unique up to natural isomorphism:

$$\begin{tikzcd} \mathcal{A} \arrow[r, tail] \arrow[dr, "F"'] & \mathsf{Pro}\mathcal{A} \arrow[d, dashed, "\overline{F}"] \\ & \mathcal{B} \end{tikzcd}$$

If $\mathcal{A}$ is finitely complete, then $\mathsf{Pro}\mathcal{A}$ is complete and cocomplete.

▶ **Remark A.2.** A concrete construction of $\mathsf{Ind}\mathcal{A}$ is the following: let $\mathsf{Ind}\mathcal{A}$ be the full subcategory of the functor category $[\mathcal{A}^{op}, \mathbf{Set}]$ on all filtered colimits of representable functors $\mathcal{A}(-, A)\colon \mathcal{A}^{op} \to \mathbf{Set}$, and let $\mathcal{Y}\colon \mathcal{A} \to \mathsf{Ind}\mathcal{A}$ be the codomain restriction of the Yoneda embedding $A \mapsto \mathcal{A}(-, A)$. Then $\mathcal{Y}$ is an ind-completion of $\mathcal{A}$. Analogously, one obtains the pro-completion as the dual Yoneda embedding $\mathcal{Y}^{op}\colon \mathcal{A} \to \mathsf{Pro}\mathcal{A}$, $A \mapsto \mathcal{A}(A, -)$. Note that $\mathsf{Pro}\mathcal{A} = (\mathsf{Ind}\mathcal{A}^{op})^{op}$.

▶ **Theorem A.3.** *Given a small finitely complete and cocomplete category $\mathcal{A}$, there is an adjunction $F \dashv U\colon \mathsf{Pro}\mathcal{A} \to \mathsf{Ind}\mathcal{A}$ such that $\mathcal{Y}^{op} = F \circ \mathcal{Y}$ and $\mathcal{Y} = U \circ \mathcal{Y}^{op}$:*

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "\mathcal{Y}"'] \arrow[dr, "\mathcal{Y}^{op}"] & \\ \mathsf{Ind}\mathcal{A} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathsf{Pro}\mathcal{A} \arrow[ll, bend left, "U"] \end{tikzcd}$$

Proof. F and U are the unique extensions of $\mathcal{Y}^{op}$ and $\mathcal{Y}$ preserving filtered colimits and cofiltered limits, respectively. Since $\mathsf{Ind}\mathcal{A}$ consists of filtered colimits of representable functors

$\mathcal{Y}(a) = \mathcal{A}(-,a)$ and similarly for $\mathsf{Pro}\mathcal{A}$, we have

$$\begin{aligned}
\mathsf{Pro}\mathcal{A}(F \operatorname*{Colim}_a \mathcal{Y}a, \operatorname*{Lim}_b \mathcal{Y}^{op}b) &\cong \operatorname*{Lim}_b \mathsf{Pro}\mathcal{A}(\operatorname{Colim} \mathcal{Y}^{op}a, \mathcal{Y}^{op}b) \\
&\cong \operatorname*{Lim}_a \operatorname*{Lim}_b \mathsf{Pro}\mathcal{A}(\mathcal{Y}^{op}a, \mathcal{Y}^{op}b) \\
&\cong \operatorname*{Lim}_a \operatorname*{Lim}_b \mathcal{A}(a,b) \\
&\cong \operatorname*{Lim}_a \operatorname*{Lim}_b \mathsf{Ind}\mathcal{A}(\mathcal{Y}a, \mathcal{Y}b) \\
&\cong \operatorname*{Lim}_b \mathsf{Ind}\mathcal{A}(\operatorname*{Colim}_a \mathcal{Y}a, \mathcal{Y}^{op}b) \\
&\cong \mathsf{Ind}\mathcal{A}(\operatorname*{Colim}_a \mathcal{Y}a, U \operatorname*{Lim}_b \mathcal{Y}^{op}b).
\end{aligned}$$

◀

B Proofs

Proof of Proposition 2.7

Let $\mathbf{Aut}_0\Sigma$ and $\mathbf{Aut}_{0,lf}\Sigma$ denote the categories of T^0_Σ-coalgebras and locally finite T^0_Σ-coalgebras, respectively. The functor $|T_\Sigma|\colon \mathcal{C} \to \mathbf{Set}$ is naturally isomorphic to $T^0_\Sigma \circ |-|$, so the adjunction $\Phi \dashv |-|\colon \mathcal{C} \to \mathbf{Set}$ induces an adjunction $\mathbf{Aut}\Phi \dashv \mathbf{Aut}|-|\colon \mathbf{Aut}\Sigma \to \mathbf{Aut}_0\Sigma$ by [9, Corollary 2.15]. The right adjoint $\mathbf{Aut}|-|$ maps an automaton (Q,γ) in $\mathcal{C}$ to its underlying automaton $(|Q|,|\gamma|)$ in $\mathbf{Set}$, and the left adjoint $\mathbf{Aut}\Phi$ maps an automaton (Q_0,γ_0) in $\mathbf{Set}$ to an automaton in $\mathcal{C}$ with carrier ΦQ_0. Since $\mathcal{C}$ is locally finite, the adjunction restricts to one between the full subcategories $\mathbf{Aut}_{lf}$ and $\mathbf{Aut}_{0,lf}$ of locally finite Σ-automata. Since the restricted right adjoint $\mathbf{Aut}|-|\colon \mathbf{Aut}_{lf}\Sigma \to \mathbf{Aut}_{0,lf}\Sigma$ preserves limits, it maps the terminal locally finite T_Σ-coalgeba ρT_Σ to the terminal locally finite T^0_Σ-coalgebra ρT^0_Σ, i.e. to the automaton of regular languages.

Proof of Lemma 3.4

1. For all $P \in \mathbf{LPV}_\Sigma$, the set $f_\sharp(P)$ forms a local pseudovariety of $\mathcal{D}$-monoids over Δ. Indeed, closure under quotients is obvious. For closure under subdirect products let $e_i\colon \Psi\Delta^* \twoheadrightarrow N_i$ $(i=1,2)$ be two Δ-generated $\mathcal{D}$-monoids in $f_\sharp(P)$, that is, $e_i f = g_i e_{M_i}$ for some $M_i \in P$ and morphisms g_i. We may assume that $M := M_1 = M_2$ – otherwise replace M_1 and M_2 by their subdirect product $M_1 \vee M_2$. Hence the left diagram below commutes. By the fill-in property, there exists a unique morphism h from M to the subdirect product $N_1 \vee N_2$ of N_1 and N_2 such that the right diagram below commutes.

$$\begin{tikzcd}
\Psi\Sigma^* \arrow[r, "f"] \arrow[d, two heads, "e_M"'] & \Psi\Delta^* \arrow[d, "\langle e_1,e_2\rangle"] \\
M \arrow[r, "\langle g_1,g_2\rangle"'] & N_1 \times N_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi\Sigma^* \arrow[r, "f"] \arrow[d, two heads, "e_M"'] & \Psi\Delta^* \arrow[d, two heads, "\mathrm{Im}\langle e_1,e_2\rangle"'] \arrow[dr, bend left, "\langle e_1,e_2\rangle"] & \\
M \arrow[r, dashed, "h"] \arrow[rr, bend right, "\langle g_1,g_2\rangle"'] & N_1 \vee N_2 \arrow[r, tail] & N_1 \times N_2
\end{tikzcd}$$

   Hence $N_1 \vee N_2$ lies in $f_\sharp(M)$.
2. $f_\sharp$ is clearly order-preserving, i.e. $P \subseteq P'$ implies $f_\sharp(P) \subseteq f_\sharp(P')$.
3. It remains to show the functoriality, i.e. $id_\sharp = id$ and $(gf)_\sharp = g_\sharp f_\sharp$ for any two $\mathcal{D}$-monoid morphisms $f\colon \Psi\Sigma^* \to \Psi\Delta^*$ and $g\colon \Psi\Delta^* \to \Psi\Gamma^*$. The first statement follows from the

closure of local pseudovarieties under quotients. For the second one let $P \in \mathbf{LPV}_\Sigma$ and suppose that $K \in g_\sharp f_\sharp(P)$. Hence there exist finite $\mathcal{D}$-monoids $M \in P$ and $N \in f_\sharp(P)$ and $\mathcal{D}$-monoid morphisms making the diagram below commute.

$$\begin{tikzcd}
\Psi\Sigma^* \arrow[r, "f"] \arrow[d, two heads] & \Psi\Delta^* \arrow[r, "g"] \arrow[d, two heads] & \Psi\Gamma^* \arrow[d, two heads] \\
M \arrow[r] & N \arrow[r] & K
\end{tikzcd}$$

This implies $K \in (gf)_\sharp(P)$. On the other hand, suppose that $K \in (gf)_\sharp(P)$, i.e. there exists some $M \in P$ and a $\mathcal{D}$-monoid morphism $h\colon M \to K$ such that the left diagram below commutes. Consider the factorisation of $e_K \circ g\colon \Psi\Delta^* \to \Psi\Gamma^* \twoheadrightarrow K$ in the right diagram:

$$\begin{tikzcd}
\Psi\Sigma^* \arrow[r, "f"] \arrow[d, two heads] & \Psi\Delta^* \arrow[r, "g"] & \Psi\Gamma^* \arrow[d, two heads, "e_K"] \\
M \arrow[rr, "h"'] & & K
\end{tikzcd}
\qquad
\begin{tikzcd}
\Psi\Sigma^* \arrow[r, "f"] \arrow[d, two heads] & \Psi\Delta^* \arrow[r, "g"] \arrow[d, two heads] & \Psi\Gamma^* \arrow[d, two heads, "e_K"] \\
M \arrow[r, dashed] \arrow[rr, bend right, "h"'] & N \arrow[r, tail] & K
\end{tikzcd}$$

By the fill-in property h factors through the submonoid N of the finite monoid K. Hence $N \in f_\sharp(P)$ and $K \in (g_\sharp f_\sharp)(P)$.

Proof of Theorem 4.5

▶ **Lemma B.1.** *Every profinite $\mathcal{D}$-monoid is the cofiltered limit of its finite quotients.*

Proof. Since the category $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$ is the pro-completion of finite $\mathcal{D}$-monoids, every profinite $\mathcal{D}$-monoid is the limit of its canonical cofiltered diagram

$$(M \downarrow \mathbf{Mon}_f\mathcal{D}) \xrightarrow{Q} \mathbf{Mon}_f\mathcal{D} \overset{i}{\rightarrowtail} \mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$$

where $(M \downarrow \mathbf{Mon}_f\mathcal{D})$ is the comma category from M to the category of finite $\mathcal{D}$-monoids, and Q is the projection functor. However, given this canonical diagram, we can always factor every morphism $M \to N$ for $N \in \mathbf{Mon}_f\mathcal{D}$ into a surjective morphism and an embedding:

$$\begin{tikzcd}
 & M \arrow[dl, two heads, "\mathrm{Im}(f)"'] \arrow[dr, two heads, "\mathrm{Im}(g)"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
f[M] \arrow[rr, dashed] \arrow[d, tail] & & g[M] \arrow[d, tail] \\
N \arrow[rr, "h"'] & & N'
\end{tikzcd}$$

This diagram consisting of all finite quotients is also cofiltered, since $i \circ Q$ is cofilterd. Then, it is easy to see that M with $\{\mathrm{Im}(f)\colon M \to f[M]\}_{f\in(M\downarrow\mathbf{Mon}_f\mathcal{D})}$ is a cofiltered limit. ◀

Proof of Theorem 4.5. (a) Let $e_M\colon \widehat{\Psi\Sigma^*} \twoheadrightarrow M$ be a profinite Σ-generated $\mathcal{D}$-monoid, and suppose that K is a finite quotient of M. Note that K is finitely copresentable in $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$,

so since $\widehat{\Psi\Sigma^*}$ is the limit of all finite quotients of $\Psi\Sigma^*$, we see that ee_M factors through some limit projection π_N:

$$\begin{tikzcd}
\Psi\Sigma^* \arrow[r] \arrow[dr] & \widehat{\Psi\Sigma^*} \arrow[r, two heads, "e_M"] \arrow[d, two heads, "\pi_N"'] & M \arrow[d, two heads, "e"] \\
& N \arrow[r, two heads, "f"'] & K
\end{tikzcd}$$

Therefore K is a Σ-generated $\mathcal{D}$-monoid. It now immediately follows that the set $\mathcal{V}_M$ of finite quotients of M forms a local pseudovariety over Σ. Clearly, the construction $M \mapsto \mathcal{V}_M$ is order-preserving, and it is injective by Lemma B.1.

(b) Conversely, we can view every local pseudovariety $P \in \mathbf{LPV}_\Sigma$ as a diagram in $\mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$ defined by

$$P \xhookrightarrow{i_P} \mathsf{Quo}_f(\Psi\Sigma^*) \xrightarrow{Q} \mathsf{Pro}\mathbf{Mon}_f\mathcal{D}$$

where i_P is the full inclusion and Q is the projection functor mapping $\Psi\Sigma^* \twoheadrightarrow M$ to M and $f\colon M \twoheadrightarrow M'$ in $\mathsf{Quo}_f(\Psi\Sigma^*)$ to f.

Note that each $M \in P$ with the discrete topology is a non-empty compact Hausdorff space. Then $M_P := \mathrm{Lim}(Q \circ i_P)$ is a profinite $\mathcal{D}$-monoid where each limit projection π_M is surjective by Lemma 4.3. Suppose that $P \subseteq P'$. Then there exists a mediating morphism from $M_{P'}$ to M_P, since the projections $M_{P'} \xrightarrow{\pi_M} M$ for $M \in P$ form a cone over $Q \circ i_P$. This mediating morphism is surjective, because every π_M is surjective. In particular, taking $P' = \mathsf{Quo}_f(\Psi\Sigma^*)$ we get a surjective morphism $\widehat{\Psi\Sigma^*} \twoheadrightarrow M_P$. (Recall that $\widehat{\Psi\Sigma^*}$ is by definition the limit of all finite quotients of $\Psi\Sigma^*$.)

(c) To show that the two construction of (a) and (b) are mutually inverse, we need to prove that, given $P \in \mathbf{LPV}_\Sigma$, every finite quotient $e_M\colon M_P = \mathrm{Lim}(i_P \circ Q) \twoheadrightarrow M$ is contained in P. Since M is finitely copresentable, the morphism e_M factors through some $N \in P$, so M must be a quotient of N; that is, $M \in P$. We conclude the construction $P \mapsto M_P$ is surjective. It is also order-preserving by the argument given in (b).

(d) The second part of theorem follows by a straighforward use of universal properties. ◀

Proof of Theorem 4.10

The proof proceeds through several lemmas.

▶ **Lemma B.2.** *Given a profinite equational theory $\mathcal{T}$ of $\mathcal{D}$-monoids, the class $\mathcal{V}$ associated to $\mathcal{T}$ forms a pseudovariety of $\mathcal{D}$-monoids.*

Proof. We need to show closure under quotients, submonoids and finite products. To this end, let $M \in \mathcal{V}$ and also finitely many $M_i \in \mathcal{V}$ be given. In the first two cases below, f refers to a morphism from $\widehat{\Psi\Sigma^*}$ to a quotient and a submonoid of M respectively. For the last case, f is a morphism to the finite product $\prod_i M_i$. See following diagrams for references.

Quotients: Given a quotient N of M with $e\colon M \twoheadrightarrow N$, since free algebras $\widehat{\Psi\Sigma^*}$ are projective there exists h with $f = eh$. By assumption h factors through e_Σ via some $\overline{h}$. Hence f factors through e_Σ via $e\overline{h}$.

Submonoids: Given a submonoid N of M, the composite mf factors through e_Σ by assumption. By the fill-in property, there is a morphism $h\colon F\Sigma \to N$ such that Diagram 4 commutes.

$$\begin{tikzcd} \widehat{\Psi\Sigma^*} \arrow[r, two heads, "e_\Sigma"] \arrow[d, "f"'] \arrow[dr, "h"] & F\Sigma \arrow[d, "\overline{h}"] \\ N & M \arrow[l, two heads, "e"] \end{tikzcd}$$

Diagram 3 Quotients

$$\begin{tikzcd} \widehat{\Psi\Sigma^*} \arrow[r, two heads, "e_\Sigma"] \arrow[d, "f"'] & F\Sigma \arrow[d, "\overline{mf}"] \arrow[dl, dashed, "h"'] \\ N \arrow[r, tail, "m"'] & M \end{tikzcd}$$

Diagram 4 Submonoids

$$\begin{tikzcd} \widehat{\Psi\Sigma^*} \arrow[r, two heads, "e_\Sigma"] \arrow[d, "f"'] & F\Sigma \arrow[d, "\overline{\pi_i f}"] \arrow[dl, dashed, "h"'] \\ \prod_i M_i \arrow[r, "\pi_i"'] & M_i \end{tikzcd}$$

Diagram 5 Finite products

Finite products: Every $\pi_i f$ factors through e_Σ by assumption, so there is a mediating morphism $h := \langle \overline{\pi_i f} \rangle$ such that Diagram 5 commutes. ◀

▶ **Lemma B.3.** *Given a pseudovariety $\mathcal{V}$ of $\mathcal{D}$-monoids the corresponding morphisms $e_\Sigma \colon \widehat{\Psi\Sigma^*} \twoheadrightarrow F\Sigma$ form a profinite equational theory.*

Proof. Recall that P_Σ is the set of Σ-generated monoids in $\mathcal{V}$. Since $\mathcal{V}$ is a pseudovariety, P_Σ is closed under quotients and subdirect products, so P_Σ is a local pseudovariety over Σ. corresponding uniquely to a quotient $e_\Sigma \colon \widehat{\Psi\Sigma^*} \twoheadrightarrow F\Sigma$ of the free profinite monoid. To see that the morphisms e_Σ form a profinite equational theory, use Theorem 4.5: for every $f \colon \Psi\Sigma^* \to \Psi\Delta^*$ and every $e \colon \Psi\Delta^* \twoheadrightarrow N$ in P_Δ, the factorisation $\Psi\Sigma^* \xrightarrow{\mathrm{Im}(ef)} M \rightarrowtail N$ of ef fulfils the left-hand diagram in the Theorem where the Σ-generated monoid M of N is in P_Σ by the fact that $\mathcal{V}$ is closed under submonoids. Hence the right diagram in the Theorem also commutes for some h, so it follows that the collection $\{e_\Sigma\}_\Sigma$ forms a profinite equational theory. ◀

Using the following lemma a straightforward verification shows that the constructions $\mathcal{T} \mapsto \mathcal{V}$ and $\mathcal{V} \mapsto \mathcal{T}$ are mutually inverse.

▶ **Lemma B.4.** *Let $\mathcal{V}$ be the pseudovariety corresponding to a profinite theory $(e_\Sigma \colon \widehat{\Psi\Sigma^*} \twoheadrightarrow F\Sigma)_\Sigma$. Then $M \in \mathcal{V}$ if and only if M is a quotient of $F|M|$.*

Proof. Suppose that $M \in \mathcal{V}$. Then M is a quotient of the free $\mathcal{D}$-monoid $\Psi|M|^*$ generated by M itself, so it is also a quotient of the free profinite $\mathcal{D}$-monoid $\widehat{\Psi|M|^*}$. By assumption, the quotient map $\widehat{\Psi|M|^*} \twoheadrightarrow M$ factors though $F|M|$ via some morphism that is necessarily surjective. The other direction follows from the projectivity of $\widehat{\Psi|M|^*}$. ◀